Discrete Mathematics 311 (2011) 1658–1665

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Ore-type conditions for bipartite graphs containing hexagons ☆

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ARTICLE INFO

Article history:
Received 20 August 2010
Received in revised form 28 March 2011
Accepted 29 March 2011
Available online 22 April 2011

Keywords:
Bipartite graph
Hexagon
Ore-type
Cycle

ABSTRACT

Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 3k$, where $k > 0$. In this paper it is proved that if $d(x) + d(y) \geq 4k - 1$ for every pair of nonadjacent vertices $x \in V_1$, $y \in V_2$, then G contains $k - 1$ independent cycles of order 6 and a path of order 6 such that all of them are independent. Furthermore, if $d(x) + d(y) \geq 4k$ for every pair of nonadjacent vertices $x \in V_1, y \in V_2$ and $k > 2$, then G contains $k - 2$ independent cycles of order 6 and a cycle of order 12 such that all of them are independent.

1. Introduction

In this paper, all graphs are finite, simple, undirected and bipartite. Any undefined notation follows that of Bondy and Murty [1]. Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2|$. We use $\delta(G)$ to denote the minimum degree in G. It is defined that $\sigma_{1,1}(G) = \min\{d(x) + d(y) \mid x \in V_1, y \in V_2, x \neq y, xy \notin E(G)\}$. The order of G is $|G|$ and its size is $e(G) = |E|$. A set of graphs is said to be independent if no two of them have any common vertex. If H is a subgraph of G, then $N_H(x) = N_G(x) \bigcap V(H)$, $d(x, H) = |N_H(x)|$. Let X and Y be two independent subgraphs of G or two disjoint subsets of $V_1 \bigcup V_2$. We define $G[X]$ to be the subgraph of G induced by X, and $e(X, Y)$ to be the number of edges between X and Y. A k-cycle is a cycle of order k and a m-path is a path of order m, denoted by C^k and P^m respectively. In particular, a quadrilateral is a cycle of order 4, and a hexagon is a cycle of order 6. For a k-cycle $C = x_1x_2 \dots x_kx_1$, x_ix_{i+1} is an edge in C and a chord of C is an edge of $G - E(C)$ which joins two vertices of C. For two independent graphs G and H, $G \cup H$ is the union of G and H without adding any edge between G and H. Let T be a simple graph and k be a positive integer, then $G \supseteq kT$ means that G contains k independent subgraphs isomorphic to T.

One of the outstanding results on cycles comes from Corrádi and Hajnal [2]. It was proved that if G is of order at least $3k$ with the minimum degree at least $2k$, then G contains k independent cycles. Wang [4] considered independent quadrilaterals in bipartite graph and showed if $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = 2k$ and $\delta(G) \geq k+1$, then G contains $k - 1$ independent quadrilaterals and a 4-path such that they are independent. Yan [7] improved this result by replacing $\delta(G) \geq k + 1$ with ore condition $\sigma_{1,1}(G) \geq 2k + 1$. Furthermore, Li [3] showed that if $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = 2k$ and $\sigma_{1,1}(G) \geq 2k + 1$, then G contains $k - 2$ independent quadrilaterals and a cycle of order 8 such that all of them are independent.

As for hexagons, Wang [5] considered independent hexagons in bipartite graph and gave the following result.

☆ This work was supported by grants 60873207, 60673059 and 10631070 from NSFC and the Taishan Scholar Fund from Shandong Province, China. It was also partially supported by Research Grant from Ningxia University under grant number: (E)NDZR09-1.

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2011.03.029

Theorem 1.1 ([5]). *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 3k$, where k is a positive integer. If the minimum degree $\delta(G) \geq 2k + 1$, then G contains k independent hexagons such that each of them has at least two chords.*

Recently, Zhu and Hao [8] lowered the minimum degree of G and got the following result.

Theorem 1.2 ([8]). *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 3k$, where k is a positive integer. If the minimum degree $\delta(G) \geq 2k - 1$, then G contains at least $k - 1$ independent hexagons.*

In this paper, we consider ore-type conditions that ensure G contains hexagons. We show the following two results.

Theorem 1.3. *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 3k$, where k is a positive integer. If $\sigma_{1,1}(G) \geq 4k - 1$, then G contains $k - 1$ independent hexagons and a 6-path such that all of them are independent.*

Theorem 1.4. *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 3k$, where $k > 2$ is a positive integer. If $\sigma_{1,1}(G) \geq 4k$, then G contains $k - 2$ independent hexagons and a 12-cycle such that all of them are independent.*

The structure of the paper is as follows. First we will show some useful lemmas. In the next section, we will prove Theorem 1.3. Finally, we will show Theorem 1.4.

2. Lemmas

In this section, $G = (V_1, V_2; E)$ is a bipartite graph.

Lemma 2.1. *Let $C = x_1y_1x_2y_2x_3y_3x_1$ be a hexagon and x, y be two distinct vertices of G not on C with $x_1 \in V_1$, $x \in V_1$, $y \in V_2$. If $d(x, C) + d(y, C) \geq 5$, then $G[V(C) \bigcup \{x, y\}]$ contains a hexagon C' and an edge e' such that C' and e' are independent and e' is incident with x. Furthermore, $G[V(C) \bigcup \{x, y\}]$ also contains a hexagon C'' and an edge e'' such that C'' and e'' are independent and e'' is incident with y.*

Proof. We will first prove the former part. Suppose $d(x, C) = 3$. This implies $d(y, C) \geq 2$, without loss of generality (denoted by w.l.o.g.), say $x_1y \in E$, $x_2y \in E$. Therefore, $G[V(C) \bigcup \{x, y\}]$ contains a hexagon $yx_1y_3x_3y_2x_2y$ and an edge xy_1 such that they are independent. Hence, $d(x, C) \leq 2$ holds. Since $d(x, C) + d(y, C) \geq 5$, we have $d(x, C) = 2$ and $d(y, C) = 3$. W.l.o.g., say $xy_1 \in E$, $xy_2 \in E$. Now $G[V(C) \bigcup \{x, y\}]$ contains a hexagon $yx_2y_1x_1y_3x_3y$ and an edge xy_2 such that they are independent. Therefore, $G[V(C) \bigcup \{x, y\}]$ contains a hexagon C' and an edge e' such that C' and e' are independent and e' is incident with x.

With the same proof, $G[V(C) \bigcup \{x, y\}]$ contains a hexagon C'' and an edge e'' such that C'' and e'' are independent and e'' is incident with y. □

Lemma 2.2 (See [8]). *Let C be a hexagon and e_1, e_2, e_3 be three independent edges such that all of them are independent. Suppose $e(C, e_1 \bigcup e_2 \bigcup e_3) \geq 13$, then $G[V(C) \bigcup V(e_1) \bigcup V(e_2) \bigcup V(e_3)]$ contains a hexagon C' and a path P of order 6 such that C' and P are independent.*

Lemma 2.3 (See [8]). *Let P_1, P_2 be two 6-paths such that they are independent. If $e(P_1, P_2) \geq 7$, then $G[V(P_1) \bigcup V(P_2)]$ contains a hexagon.*

Lemma 2.4 (See [8]). *Let C be a hexagon and P_1, P_2 be two 6-paths in G such that all of them are independent. If $e(C, P_1 \bigcup P_2) \geq 25$, then $G[V(C) \bigcup V(P_1) \bigcup V(P_2)]$ contains two independent hexagons.*

Lemma 2.5. *Let F be a subgraph of G with $|F| = 6$. If F contains three independent edges xy, ab, uv with $\{x, u, a\} \subseteq V_1$ and $e(G[V(F)]) \geq 6$, then $G[V(F)]$ contains a 6-path.*

Proof. If there exist two edges $\{e_1, e_2\} \subseteq \{xy, uv, ab\}$, such that $e(e_1, e_2) = 2$. W.l.o.g., say $e(xy, uv) = 2$. Since $e(G[V(F)]) \geq 6$, $e(ab, yxuv) \geq 1$. Since $xyuvx$ is a quadrilateral in G, we have $G[V(F)]$ contains a 6-path. Therefore, $e(e_1, e_2) \leq 1$ for any two edges $\{e_1, e_2\} \subseteq \{xy, uv, ab\}$. Since $|E(G[V(F)]) - \{xy, uv, ab\}| \geq 6 - 3 = 3$, it follows that $e(xy, uv) = e(xy, ab) = e(uv, ab) = 1$. W.l.o.g., say $xv \in E$. If $ub \in E$, then $G[V(F)]$ contains a 6-path. Hence we may assume $ub \notin E$. Note that $e(ab, uv) = 1$, we have $va \in E$. Since $e(xy, ab) = 1$, either $xb \in E$ or $ya \in E$. In the former case, $yxbavu$ is a 6-path in $G[V(F)]$; in the latter case, $uvxyab$ is a 6-path in $G[V(F)]$. □

The following lemma is of great importance in our proof.

Lemma 2.6. *Let P_1, P_2 be two 6-paths in G such that P_1 and P_2 are independent. If $G[V(P_1) \bigcup V(P_2)] \not\supseteq C^6$, then $V(P_1 \bigcup P_2)$ can be divided into six pairs of nonadjacent vertices such that each pair of vertices have a vertex in V_1 and a vertex in V_2.*

Proof. Let $P_1 = x_1y_1x_2y_2x_3y_3$, $P_2 = a_1b_1a_2b_2a_3b_3$ with $\{a_1, x_1\} \subseteq V_1$. In the following we will first prove that $\{x_1, x_2, x_3, b_1, b_2, b_3\}$ can be divided into three pairs of nonadjacent vertices such that each pair of vertices have a vertex in V_1 and a vertex in V_2.

If $e(x_1x_2x_3, P_2) \leq 2$, then obviously $\{x_1, x_2, x_3, b_1, b_2, b_3\}$ can be divided into three pairs of nonadjacent vertices such that each pair of vertices have a vertex in V_1 and a vertex in V_2. Hence $e(x_1x_2x_3, P_2) \geq 3$. Since $G[V(P_1) \bigcup V(P_2)]$ does not contain a hexagon, we have $\{x_1b_1, x_1b_3\} \not\subseteq E$ and $d(x_1, P_2) \leq 2$. With the same proof, $d(x_2, P_2) \leq 2$ and $d(x_3, P_2) \leq 2$. The following proof is divided into three cases: $d(x_2, P_2) = 0$, $d(x_2, P_2) = 1$ or $d(x_2, P_2) = 2$.

Case 1. $d(x_2, P_2) = 0$.

Note that $e(x_1x_2x_3, P_2) \geq 3$, then $e(x_1x_3, P_2) \geq 3$. Since $d(x_1, P_2) \leq 2$ and $d(x_3, P_2) \leq 2$, it follows that $d(x_1, P_2) > 0$, $d(x_3, P_2) > 0$. Let $x_1b_i \in E$, $x_3b_j \in E$, $i, j \in \{1, 2, 3\}$. If $i = j$, then $G[V(P_1) - y_3 + b_i]$ contains a hexagon, a contradiction. Thus, $i \neq j$. Since $G[V(P_1) \bigcup V(P_2)] \not\supseteq C^6$, we have $x_3b_i \notin E$ and $x_1b_j \notin E$. Denote $p \in \{1, 2, 3\}$ with $\{i, j, p\} = \{1, 2, 3\}$. Now $\{x_1, b_j\}$, $\{x_2, b_p\}$ and $\{x_3, b_i\}$ are three pairs of nonadjacent vertices.

Case 2. $d(x_2, P_2) = 1$.

Suppose $x_2b_2 \in E$. Then $e(x_1x_3, b_1b_3) = 0$, otherwise $G[V(P_1) \bigcup V(P_2)]$ contains a hexagon. Since $e(x_1x_2x_3, P_2) \geq 3$, $e(x_1x_3, b_2) = 2$. Now $G[V(P_1) \bigcup V(P_2)]$ contains a hexagon $x_1y_1x_2y_2x_3b_2\, x_1$, a contradiction. Thus we have either $x_2b_1 \in E$ or $x_2b_3 \in E$. In each case, $e(x_1x_3, b_2) = 0$, $e(x_1x_3, b_1) \leq 1$ and $e(x_1x_3, b_3) \leq 1$. Since $e(x_1x_2x_3, P_2) \geq 3$, we have $e(x_1x_3, b_1b_3) = 2$. Since $G[V(P_1) \bigcup V(P_2)] \not\supseteq C^6$, w.l.o.g., say $x_1b_1 \in E$ and $x_3b_3 \in E$. Hence $\{x_1, b_3\}$, $\{x_2, b_2\}$ and $\{x_3, b_1\}$ are three pairs of nonadjacent vertices.

Case 3. $d(x_2, P_2) = 2$.

In this case, either $\{x_2b_1, x_2b_2\} \subseteq E$ or $\{x_2b_2, x_2b_3\} \subseteq E$. If $\{x_2b_1, x_2b_2\} \subseteq E$ holds, then $x_1b_1 \notin E$, $x_3b_2 \notin E$ for otherwise $G[V(P_1) \bigcup V(P_2)]$ contains a hexagon. Thus $\{x_1, b_1\}$, $\{x_3, b_2\}$ and $\{x_2, b_3\}$ are three pairs of nonadjacent vertices. The proof is similar when $\{x_2b_2, x_2b_3\} \subseteq E$.

Hence $\{x_1, x_2, x_3, b_1, b_2, b_3\}$ can be divided into three pairs of nonadjacent vertices such that each pair of vertices have a vertex in V_1 and a vertex in V_2 in each case. With the same proof, $\{y_1, y_2, y_3, a_1, a_2, a_3\}$ can be divided into three pairs of nonadjacent vertices such that each pair of vertices have a vertex in V_1 and a vertex in V_2. Thus $V(P_1 \bigcup P_2)$ can be divided into six pairs of nonadjacent vertices such that each pair of vertices have a vertex in V_1 and a vertex in V_2. $\square$

Lemma 2.7 (*See [6]*)**.** *Let C be a cycle of order $2k$ and P be a path with two endvertices $u \in V_1$, $v \in V_2$, where $k > 0$. If C and P are independent and $d(u, C) + d(v, C) \geq k + 1$, then $G[V(P) \bigcup V(C)]$ contains a cycle C' satisfying $V(C') = V(P \bigcup C)$.*

Lemma 2.8. *Let C_1, C_2 be two independent hexagons in G. If $e(C_1, C_2) \geq 10$, then $G[V(C_1) \bigcup V(C_2)]$ contains a cycle of order 12.*

Proof. Denote $C_1 = x_1y_1x_2y_2x_3y_3x_1$, $C_2 = a_1b_1a_2b_2a_3b_3a_1$ with $\{x_1, a_1\} \subseteq V_1$. Since $e(C_1, C_2) \geq 10$, there exists $i \in \{1, 2, 3\}$ such that $d(x_i, C_2) + d(y_i, C_2) \geq 4$. W.l.o.g., say $i = 1$. Since $x_1y_3x_3y_2x_2y_1$ is a 6-path with two endvertices x_1, y_1, $G[V(C_1) \bigcup V(C_2)]$ contains a cycle of order 12 from Lemma 2.7. $\square$

Lemma 2.9. *Let C_1, C_2 be two independent hexagons in G. If $e(C_1, C_2) \geq 7$ and $G[V(C_1) \bigcup V(C_2)] \not\supseteq C^{12}$, then either $e(V(C_1) \bigcap V_1, C_2) = e(C_1, C_2)$ or $e(V(C_1) \bigcap V_2, C_2) = e(C_1, C_2)$.*

Proof. Let $C_1 = x_1y_1x_2y_2x_3y_3x_1$ and $C_2 = a_1b_1a_2b_2a_3b_3a_1$ with $\{x_1, a_1\} \subseteq V_1$. For simplicity, denote $e_1 = x_1y_1$, $e_2 = x_2y_2$, $e_3 = x_3y_3$, $l_1 = a_1b_1$, $l_2 = a_2b_2$, $l_3 = a_3b_3$. Obviously, $e(C_1, C_2) = \sum_{i=1}^{3} e(e_i, C_2)$. If there exist $i, j \in \{1, 2, 3\}$ such that $e(e_i, l_j) = 2$, then $G[V(C_1) \bigcup V(C_2)] \supseteq C^{12}$, a contradiction. Thus,

$$e(e_i, l_j) \leq 1 \quad i, j = 1, 2, 3.$$

Hence we have $e(e_i, C_2) \leq 3$ for all $i \in \{1, 2, 3\}$. Since $e(C_1, C_2) \geq 7$, w.l.o.g., say $e(e_1, C_2) = 3$. Furthermore, $e(e_1, l_1) = e(e_1, l_2) = e(e_1, l_3) = 1$. Note that $e(e_1, l_1) = 1$, by symmetry, say $x_1b_1 \in E$. Now $y_1a_2 \notin E$ for otherwise $G[V(C_1) \bigcup V(C_2)] \supseteq C^{12}$. Since $e(e_1, l_2) = 1$, it follows that $x_1b_2 \in E$. Similarly, $x_1b_3 \in E$ and therefore $d(x_1, C_2) = 3$. This implies $d(y_1, C_2) = 0$ and $d(y_3, C_2) = 0$ for otherwise $G[V(C_1) \bigcup V(C_2)] \supseteq C^{12}$.

Suppose $e(y_2, C_2) > 0$. Since $e(e_2e_3, C_2) = e(C_1, C_2) - d(x_1, C_2) - d(y_1, C_2) \geq 4$ and $e(e_2, C_2) \leq 3$, we have $e(e_3, C_2) \geq 1$. This implies $d(x_3, C_2) \geq 1$. By symmetry, say $x_3b_3 \in E$. If either $y_2a_3 \in E$ or $y_2a_1 \in E$, then $G[V(C_1) \bigcup V(C_2)] \supseteq C^{12}$, a contradiction. Thus $y_2a_2 \in E$ and $d(y_2, C_2) = 1$. If $e(x_2x_3, b_1b_2) > 0$, then $G[V(C_1) \bigcup V(C_2)] \supseteq C^{12}$, a contradiction. Hence $e(x_2x_3, b_1b_2) = 0$. Therefore, $e(C_1, C_2) \leq 6$, contradicting the condition $e(C_1, C_2) \geq 7$. Thus we may assume $e(y_2, C_2) = 0$. Now $e(V(C_1) \bigcap V_2, C_2) = 0$ and therefore $e(V(C_1) \bigcap V_1, C_2) = e(C_1, C_2)$. $\square$

3. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3 by contradiction. Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 3k$ and $\sigma_{1,1}(G) \geq 4k - 1$, where k is a positive integer. Suppose on the contrary that $G \not\supseteq (k-1)C^6 \bigcup P^6$. Let G be a maximal counterexample, that is, $G + xy \supseteq (k-1)C^6 \bigcup P^6$ for any $xy \notin E$. Thus G either contains $k-2$ independent hexagons and two independent 6-paths such that all of them are independent or contains $k-1$ independent hexagons.

Claim 1. $G \supseteq (k-1)C^6$.

Proof. If not, then G contains $k-2$ independent hexagons and two independent 6-paths such that all of them are independent. Let $Q_1, Q_2, \ldots, Q_{k-2}$ be the $k-2$ hexagons and P_1, P_2 be the two 6-paths such that all of them are independent. Denote $H = \bigcup_{i=1}^{k-2} Q_i$. Since $G[V(P_1) \bigcup V(P_2)] \not\supseteq C^6$, by Lemmas 2.3 and 2.6, it follows that $e(P_1, P_2) \leq 6$ and $V(P_1 \bigcup P_2)$

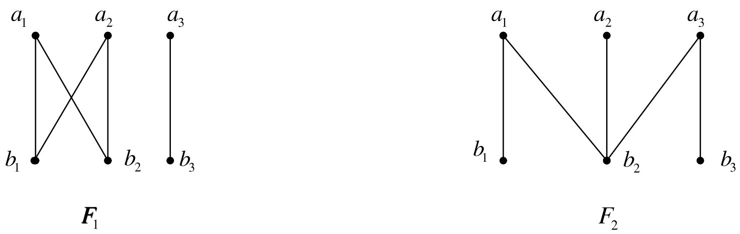

Fig. 1. F is isomorphic to F_1 or F_2.

can be divided into six pairs of nonadjacent vertices. Since $G[V(P_1)] \not\supseteq C^6$ and $G[V(P_2)] \not\supseteq C^6$, we have $e(P_1) \le 7, e(P_2) \le 7$. Therefore, $\sum_{x \in V(P_1 \bigcup P_2)} d(x, P_1 \bigcup P_2) \le 14 + 14 + 12 = 40$. Hence,

$$\sum_{x \in V(P_1 \bigcup P_2)} d(x, H) \ge 6(4k-1) - 40 = 24(k-2) + 2.$$

Thus, there exists a hexagon $Q_i \subseteq H$, say Q_1, such that $e(P_1 \bigcup P_2, Q_1) \ge 25$. By Lemma 2.4, $G[V(P_1) \bigcup V(P_2) \bigcup V(Q_1)]$ contains two independent hexagons. Therefore, $G \supseteq (k-1)C^6$, contradicting our assumption. □

By Claim 1, $G \supseteq (k-1)C^6$, denoted by $Q_1, Q_2, \ldots, Q_{k-1}$. Let $H = \bigcup_{i=1}^{k-1} Q_i$, $F = G - H$. Obviously, $F \not\supseteq P^6$. Now we choose $Q_1, Q_2, \ldots, Q_{k-1}$ in G such that

the number of independent edges of F is maximal. (1)

Claim 2. *F contains at least one independent edge.*

Proof. Suppose on the contrary that F contains no edge. Choose $x \in V(F) \bigcap V_1$, $y \in V(F) \bigcap V_2$. So $d(x, F) + d(y, F) = 0$. Since $xy \notin E$, $d(x, H) + d(y, H) \ge 4k-1 = 4(k-1)+3$. Hence there exists a hexagon $Q_i \subseteq H$, such that $d(x, Q_i) + d(y, Q_i) \ge 5$. By Lemma 2.1, $G[V(Q_i) \bigcup \{x, y\}]$ contains a hexagon Q_i' and an edge e such that they are independent. Replacing Q_i with Q_i', we get a contradiction with (1). □

In the following, let a_1b_1 be an edge in F with $a_1 \in V_1$.

Claim 3. *F contains at least two independent edges.*

Proof. Suppose on the contrary that F does not contain two independent edges. Choose $u \in V(F - a_1 - b_1) \bigcap V_1$, $v \in V(F - a_1 - b_1) \bigcap V_2$. So $d(u, F) + d(v, F) \le 1$ and $uv \notin E$. Therefore, $d(u, H) + d(v, H) \ge 4k - 2 = 4(k-1) + 2$. Hence there exists a hexagon $Q_i \subseteq H$, such that $d(u, Q_i) + d(v, Q_i) \ge 5$. By Lemma 2.1, $G[V(Q_i) \bigcup \{u, v\}]$ contains a hexagon Q_i' and an edge e such that they are independent. Replacing Q_i with Q_i', we get a contradiction with (1). □

In the following, let a_1b_1, a_2b_2 be two independent edges in F and $\{a_3, b_3\} = V(F - a_1b_1 - a_2b_2)$, where $\{a_1, a_2, a_3\} \subseteq V_1$.

Claim 4. *F contains three independent edges.*

Proof. Suppose on the contrary that F contains only two independent edges. So $d(a_3, F) + d(b_3, F) \le 2$ and $a_3b_3 \notin E$. Therefore, $d(a_3, H) + d(b_3, H) \ge 4k-3 = 4(k-1)+1$. Hence there exists a hexagon $Q_i \subseteq H$, such that $d(a_3, Q_i) + d(b_3, Q_i) \ge 5$. By Lemma 2.1, $G[V(Q_i) \bigcup \{a_3, b_3\}]$ contains a hexagon Q_i' and an edge e such that they are independent. Replacing Q_i with Q_i', we get a contradiction with (1). □

Now we will complete our proof. Denote the three independent edges in F by a_1b_1, a_2b_2 and a_3b_3 with $\{a_1, a_2, a_3\} \subseteq V_1$. Since $F \not\supseteq P^6$, we have $e(F) \le 5$ from Lemma 2.5.

Suppose $e(F) \le 4$. Obviously $V(F)$ can be divided into three pairs of nonadjacent vertices such that each pair of vertices have a vertex in V_1 and a vertex in V_2. Since $\sum_{x \in V(F)} d(x, F) \le 8$, we have

$$e(F, H) \ge 3(4k-1) - 8 = 12(k-1) + 1.$$

Hence there exists a hexagon $Q_i \subseteq H$, such that $e(F, Q_i) \ge 13$. By Lemma 2.2, $G[V(F) \bigcup V(Q_i)]$ contains a hexagon Q_i' and a 6-path such that they are independent. Replace Q_i with Q_i', a contradiction.

Now we may assume $e(F) = 5$. Since $F \not\supseteq P^6$, F is isomorphic to one of the following two graphs (see Fig. 1).

The following proof is divided into two cases: F is isomorphic to F_1 or F_2.

Case 1. F is isomorphic to F_2.

In this case, $d(b_1, F) + d(a_2, F) = 2$ and $a_2b_1 \notin E$. Thus, $e(a_2b_1, H) \ge 4k-3 = 4(k-1)+1$. Hence there exists a hexagon $Q_i \subseteq H$ such that $e(a_2b_1, Q_i) \ge 5$. By Lemma 2.1, $G[V(Q_i) \bigcup \{a_2, b_1\}]$ contains a hexagon Q_i' and an edge b_1x such that they are independent, where $x \in V(Q_i)$. Replacing Q_i with Q_i', G contains $k-1$ independent hexagons and a 6-path $xb_1a_1b_2a_3b_3$ such that all of them are independent, a contradiction.

Case 2. F is isomorphic to F_1.

Let $S=\{a_1,a_2,b_1,b_2\}$. So $2(d(a_3,F)+d(b_3,F))+\sum_{x\in S}d(x,F)=4+8=12$. Since $a_3b_1\notin E$, $a_3b_2\notin E$, $b_3a_1\notin E$ and $b_3a_2\notin E$, it follows that

$$2e(a_3b_3,H)+e(S,H)\geq 4(4k-1)-12=16(k-1). \tag{2}$$

Suppose there exists a hexagon $Q_i\subseteq H$, such that $2e(a_3b_3,Q_i)+e(S,Q_i)\geq 17$. Denote $Q_i=x_1y_1x_2y_2x_3y_3x_1$ with $x_1\in V_1$. If $e(a_3b_3,Q_i)\leq 4$, then $e(F,Q_i)\geq 13$ and therefore $G[V(F)\bigcup V(Q_i)]$ contains a hexagon and a 6-path such that they are independent from Lemma 2.2, a contradiction. Now we may assume $5\leq e(a_3b_3,Q_i)\leq 6$, w.l.o.g., say $d(a_3,Q_i)=3$, $d(b_3,Q_i)\geq 2$. By symmetry, let $b_3x_2\in E$, $b_3x_3\in E$. Since $a_3y_1x_1y_3x_3b_3a_3$ is a hexagon and $G[V(F)\bigcup V(Q_i)]\not\supseteq C^6\bigcup P^6$, we have $e(x_2y_2,S)=0$. With the same proof, $e(x_1y_3,S)=0$. Thus, $e(Q_i,S)=e(x_3y_1,S)+e(x_2y_2,S)+e(x_1y_3,S)\leq 4$. Therefore, $2e(a_3b_3,Q_i)+e(S,Q_i)\leq 12+4=16$, a contradiction. Hence, for each hexagon $Q_j\subseteq H$, $2e(a_3b_3,Q_j)+e(S,Q_j)\leq 16$. By (2), we have

$$2e(a_3b_3,Q_j)+e(S,Q_j)=16\quad\forall Q_j\subseteq H. \tag{3}$$

If there exists a hexagon $Q_i\subseteq H$ such that $e(a_3b_3,Q_i)\leq 3$, then $e(F,Q_i)\geq 13$ and therefore $G[V(F)\bigcup V(Q_i)]$ contains a hexagon and a 6-path such that they are independent from Lemma 2.2, a contradiction. Thus,

$$e(a_3b_3,Q_j)\geq 4\quad\forall Q_j\subseteq H. \tag{4}$$

By symmetry, we will divide the following proof into two subcases: either $d(a_3,Q_j)=3$ or $d(a_3,Q_j)=d(b_3,Q_j)=2$.

Subcase 2.1. $\exists Q_j\subseteq H$, $d(a_3,Q_j)=3$.

Denote $Q_j=x_1y_1x_2y_2x_3y_3x_1$ with $x_1\in V_1$. By (4), $d(b_3,Q_j)\geq 1$, w.l.o.g., say $b_3x_1\in E$. Since $b_3x_1y_3x_3y_2a_3b_3$ is a hexagon and $G[V(F)\bigcup V(Q_j)]\not\supseteq C^6\bigcup P^6$, it follows that $e(x_2y_1,S)=0$. With the same proof, $e(x_3y_3,S)=0$. If $d(b_3,x_2x_3)>0$, w.l.o.g., say $x_2b_3\in E$, then similarly we get $e(x_1y_1,S)=0$ and $e(y_2x_3,S)=0$. Thus $e(S,Q_j)=0$ and therefore $2e(a_3b_3,Q_j)+e(S,Q_j)\leq 12$, contradicting (3). Therefore, $d(b_3,x_2x_3)=0$ and $d(b_3,Q_j)=1$ hold. And so $2e(a_3b_3,Q_j)+e(S,Q_j)\leq 8+4=12$, contradicting (3).

Subcase 2.2. $\forall Q_j\subseteq H$, $d(a_3,Q_j)=d(b_3,Q_j)=2$.

Denote $Q_j=x_1y_1x_2y_2x_3y_3x_1$ with $x_1\in V_1$. Since $d(a_3,Q_j)=2$, w.l.o.g., say $a_3y_1\in E$ and $a_3y_2\in E$. Since $d(b_3,Q_j)=2$, we have either $b_3x_1\in E$ or $b_3x_3\in E$. By symmetry, say $b_3x_3\in E$. Since $b_3x_3y_2x_2y_1a_3b_3$ is a hexagon and $G[V(F)\bigcup V(Q_j)]\not\supseteq C^6\bigcup P^6$, it follows that $e(x_1y_3,S)=0$. With the same proof, $e(x_2y_2,S)=0$. Thus, $e(S,Q_j)=e(x_1y_3,S)+e(x_2y_2,S)+e(x_3y_1,S)\leq 4$. Hence $2e(a_3b_3,Q_j)+e(S,Q_j)\leq 8+4=12$, contradicting (3). This completes the whole proof of Theorem 1.3.

4. Proof of Theorem 1.4

Let $G=(V_1,V_2;E)$ be a bipartite graph with $|V_1|=|V_2|=3k$ and $\sigma_{1,1}(G)\geq 4k$, where $k>2$ is a positive integer. Suppose on the contrary that G does not contain $k-2$ independent hexagons and a 12-cycle such that all of them are independent. Since $\sigma_{1,1}(G)\geq 4k>4k-1$, $G\supseteq(k-1)C^6\bigcup P^6$ from Theorem 1.3. Denote the $k-1$ independent hexagons of G by $Q_1,Q_2,\ldots,Q_{k-1}$. Let $H=\bigcup_{i=1}^{k-1}Q_i$ and $P=x_1y_1x_2y_2x_3y_3$ be a 6-path in $G-H$ with $x_1\in V_1$.

Claim 5. *G contains k independent hexagons.*

Proof. Suppose on the contrary that $G\not\supseteq kC^6$. So $G[V(P)]$ does not contain a hexagon. This implies $x_1y_3\notin E$ and $d(x_1,P)+d(y_3,P)\leq 3$. Hence $d(x_1,H)+d(y_3,H)\geq 4k-3=4(k-1)+1$. Thus, there exists a hexagon $Q_i\subseteq H$ such that $d(x_1,Q_i)+d(y_3,Q_i)\geq 5$. By Lemma 2.7, $G[V(P)\bigcup V(Q_i)]$ contains a cycle of order 12. Thus, G contains $k-2$ independent hexagons and a 12-cycle such that all of them are independent, a contradiction. □

In the following, let $Q_1,Q_2,\ldots,Q_k$ be the k independent hexagons contained in G. Since $G\not\supseteq(k-2)C^6\bigcup C^{12}$, by Lemma 2.8, we have

$$e(Q_i,Q_j)\leq 9,\quad\forall i\neq j. \tag{5}$$

Now we will define a directed graph $D=(V,A)$ with $|V(D)|=k$. Each hexagon Q_i is a vertex of D. Arc set $A(D)$ is defined as follows:

$$Q_iQ_j\in A\quad\text{if and only if}\quad e\left(Q_i\bigcap V_1,Q_j\right)\geq 7$$

$$Q_jQ_i\in A\quad\text{if and only if}\quad e\left(Q_i\bigcap V_2,Q_j\right)\geq 7.$$

Claim 6. *For every $i\neq j$, if $Q_iQ_j\in A$, then $e(Q_i\bigcap V_2,Q_j)=0$. Furthermore, $Q_jQ_i\notin A$.*

Proof. Denote $Q_i=x_1y_1x_2y_2x_3y_3x_1$ and $Q_j=a_1b_1a_2b_2a_3b_3a_1$ with $\{x_1,a_1\}\subseteq V_1$. Since $Q_iQ_j\in A$, it follows that $e(x_1x_2x_3,Q_j)\geq 7$. Hence, there exists a vertex, w.l.o.g., say x_1, such that $d(x_1,Q_j)=3$. If either $e(y_1,Q_j)>0$ or $e(y_3,Q_j)>0$, then $G[V(Q_i)\bigcup V(Q_j)]$ contains a 12-cycle, a contradiction. Thus, $d(y_1,Q_j)=d(y_3,Q_j)=0$.

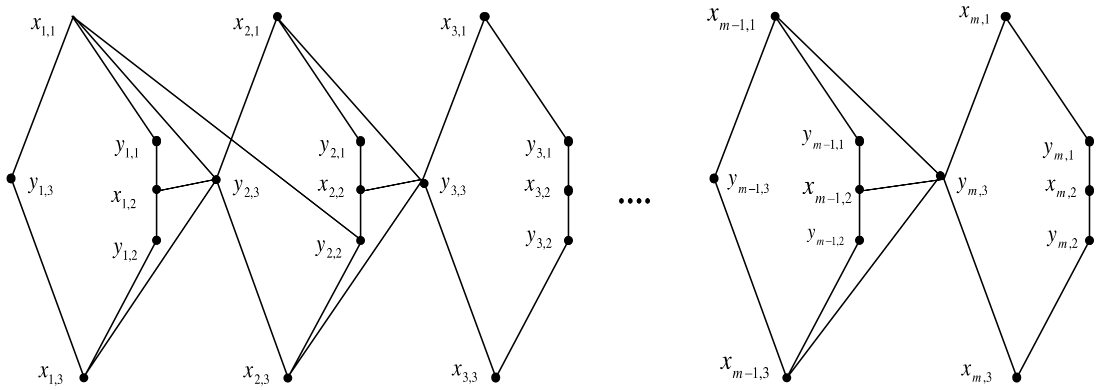

Fig. 2. Chain 1.

Since $e(x_1x_2x_3, Q_j) \geq 7$, we have $e(x_2x_3, Q_j) \geq 4$, which implies that either $d(x_2, Q_j) \geq 2$ or $d(x_3, Q_j) \geq 2$. By symmetry, say $d(x_2, Q_j) \geq 2$. Then $e(y_2, Q_j) = 0$ for otherwise $G[V(Q_i) \bigcup V(Q_j)]$ contains a 12-cycle. Thus, $e(Q_i \bigcap V_2, Q_j) = 0$ and therefore $Q_jQ_i \notin A$. □

Claim 7. *D has at least one arc.*

Proof. Suppose on the contrary that D has no arc. If there exist two independent hexagons Q_i, Q_j such that $e(Q_i, Q_j) \geq 7$, by Lemma 2.9, either $e(Q_i \bigcap V_1, Q_j) \geq 7$ or $e(Q_i \bigcap V_2, Q_j) \geq 7$. In each case, D contains an arc, a contradiction. Thus,

$$e(Q_i, Q_j) \leq 6, \quad \forall i \neq j.$$

If there exist two independent hexagons Q_i, Q_j such that $e(Q_i, Q_j) \leq 5$, then either $e(Q_i \bigcap V_1, Q_j) \leq 2$ or $e(Q_i \bigcap V_2, Q_j) \leq 2$. W.l.o.g., say $e(Q_i \bigcap V_1, Q_j) \leq 2$. Denote $Q_i = x_1y_1x_2y_2x_3y_3x_1$ and $Q_j = a_1b_1a_2b_2a_3b_3a_1$ with $\{x_1, a_1\} \subseteq V_1$. Thus, $e(x_1x_2x_3, b_1b_2b_3) \leq 2$. Therefore, the set $S = \{x_1, x_2, x_3, b_1, b_2, b_3\}$ can be divided into three pairs of nonadjacent vertices such that each pair of vertices have a vertex in V_1 and a vertex in V_2. Since $\sum_{x \in S} d(x, Q_i \bigcup Q_j) \leq 9 + 9 + 4 = 22$, we have $\sum_{x \in S} d(x, G - Q_i - Q_j) \geq 12k - 22 = 12(k-2) + 2$. Thus there exists a hexagon $Q_p \subseteq G - Q_i - Q_j$ such that $e(S, Q_p) \geq 13$. Therefore, either $e(x_1x_2x_3, Q_p) \geq 7$ or $e(b_1b_2b_3, Q_p) \geq 7$. In each case, D contains an arc, a contradiction.

Now we have $e(Q_i, Q_j) = 6$ for every $i \neq j$. If $e(Q_i \bigcap V_1, Q_j) \leq 2$ or $e(Q_i \bigcap V_2, Q_j) \leq 2$, with the same proof above, D contains an arc. Thus,

$$e\left(Q_i \bigcap V_1, Q_j\right) = e\left(Q_i \bigcap V_2, Q_j\right) = 3, \quad \forall i \neq j. \tag{6}$$

Denote $Q_1 = c_1d_1c_2d_2c_3d_3c_1$ and $Q_2 = u_1v_1u_2v_2u_3v_3u_1$ with $\{c_1, u_1\} \subseteq V_1$. Let $M = \{c_1, c_2, c_3, v_1, v_2, v_3\}$.

Suppose there exists a vertex $x \in M$ such that $d(x, M) = 3$. By symmetry, say $d(c_1, M) = 3$, then $d(c_2, Q_2) = d(c_3, Q_2) = 0$ from (6). Since $G[V(Q_1) \bigcup V(Q_2)]$ does not contain a 12-cycle, it follows that $d(d_1, Q_2) = d(d_3, Q_2) = 0$. Note that $e(d_1d_2d_3, Q_2) = 3$, thus $d(d_2, Q_2) = 3$. Denote $T = \{c_2, c_3, d_1, d_3, v_1, v_3, u_2, u_3\}$. Then $\sum_{x \in T} d(x, Q_1 \bigcup Q_2) \leq 4 + 12 + 12 = 28$. For each hexagon $Q_p \subseteq G - Q_1 - Q_2$, $e(c_2c_3d_1d_3, Q_p) \leq e(Q_1, Q_p) = 6$ and $e(v_1v_3u_2u_3, Q_p) \leq e(Q_2, Q_p) = 6$. Hence, $\sum_{x \in T} d(x, G) \leq 28 + 12(k-2) = 12k + 4$. On the other hand, since $c_2v_1 \notin E$, $c_3v_3 \notin E$, $d_1u_2 \notin E$ and $d_3u_3 \notin E$, it follows that $\sum_{x \in T} d(x, G) \geq 16k$. Therefore, $16k \leq 12k + 4$, which implies $k \leq 1$, contradicting the condition $k > 2$.

Now we may assume that $d(x, M) \leq 2$ for each vertex $x \in M$. By (6), $e(c_1c_2c_3, v_1v_2v_3) = 3$. Obviously, M can be divided into three pairs of nonadjacent vertices such that each pair of vertices have a vertex in V_1 and a vertex in V_2. This implies that $\sum_{x \in M} d(x, G) \geq 12k$. By (6), for each hexagon $Q_p \subseteq G - Q_1 - Q_2$, $e(c_1c_2c_3, Q_p) = e(v_1v_2v_3, Q_p) = 3$. Thus, $\sum_{x \in M} d(x, G) \leq 3(k-1) + 9 + 3(k-1) + 9 = 6k + 12$. Therefore, $12k \leq 6k + 12$, which implies $k \leq 2$, contradicting the condition $k > 2$. □

In the following, denote $Q_i = x_{i,1}y_{i,1}x_{i,2}y_{i,2}x_{i,3}y_{i,3}x_{i,1}$ with $x_{i,1} \in V_1$ for every $i \in \{1, 2, \ldots, k\}$. Now we will introduce a chain in G. Suppose $Q_1Q_2 \ldots Q_m$ be a directed path in D. Relabel the $m-1$ independent hexagons $Q_2, Q_3, \ldots, Q_m$ such that $d(y_{i+1,3}, Q_i) \geq d(y_{i+1,2}, Q_i) \geq d(y_{i+1,1}, Q_i)$ for each $i \in \{1, 2, \ldots, m-1\}$. Since $Q_iQ_{i+1} \in A$ for every $1 \leq i \leq m-1$, it follows that $e(y_{i+1,1}y_{i+1,2}y_{i+1,3}, Q_i) \geq 7$. Hence $d(y_{i+1,3}, Q_i) = 3$ and $d(y_{i+1,2}, Q_i) \geq 2$ for every $1 \leq i \leq m-1$. In particular, since $d(y_{2,2}, Q_1) \geq 2$, either $y_{2,2}x_{1,1} \in E$ or $y_{2,2}x_{1,3} \in E$. By symmetry, say $y_{2,2}x_{1,1} \in E$. Thus G contains a chain, denoted by Chain 1 (see Fig. 2).

Claim 8. *There is no directed cycle in D.*

Proof. Suppose on the contrary that D contains a directed cycle. W.l.o.g., let $Q_1Q_2 \ldots Q_mQ_1$ be a directed cycle in D. By Claim 6, $m \geq 3$. By the discussion above, $G[\bigcup_{i=1}^{m} V(Q_i)]$ contains Chain 1 (see Fig. 2). So $G[\bigcup_{i=1}^{m} V(Q_i)]$ contains a 12-cycle $x_{1,1}y_{2,2}x_{2,2}y_{2,1}x_{2,1}y_{3,3}x_{2,3}y_{2,3}x_{1,3}y_{1,2}\, x_{1,2}y_{1,1}x_{1,1}$ and $m-2$ independent hexagons $x_{i,1}y_{i,1}x_{i,2}y_{i,2}x_{i,3}y_{i+1,3}x_{i,1}$ $(3 \leq i \leq m$, subscript module $m)$ such that all of them are independent. This implies that $G \supseteq (k-2)C^6 \bigcup C^{12}$, a contradiction. □

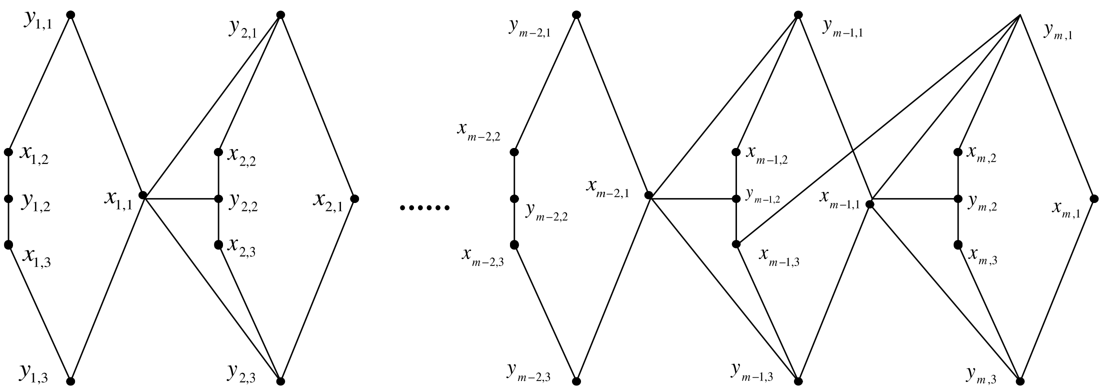

Fig. 3. Chain 2.

Let L be the longest directed path in D, w.l.o.g., say $L = Q_1Q_2 \dots Q_m$. By Claim 7, $m \geq 2$. Since L is the longest directed path in D, we have $Q_jQ_1 \notin A$ and $Q_mQ_j \notin A$ for every $m < j \leq k$. This implies that

$$e\left(Q_1 \bigcap V_2, Q_j\right) \leq 6, \qquad e\left(Q_m \bigcap V_1, Q_j\right) \leq 6 \quad \forall m < j \leq k.$$

Claim 9. *The vertex set $(Q_1 \bigcap V_2) \bigcup (Q_m \bigcap V_1)$ cannot be divided into three pairs of nonadjacent vertices such that each pair of vertices have a vertex in V_1 and a vertex in V_2.*

Proof. Denote $T = (Q_1 \bigcap V_2) \bigcup (Q_m \bigcap V_1)$. Suppose on the contrary that T can be divided into three pairs of nonadjacent vertices such that each pair of vertices have a vertex in V_1 and a vertex in V_2. So $\sum_{x \in T} d(x, G) \geq 12k$.

Since $Q_1Q_2 \in A$, we have $e(Q_1 \bigcap V_2, Q_2) = 0$ from Claim 6. For every $3 \leq j \leq m$, $Q_jQ_1 \notin A$ for otherwise D contains a directed cycle, contradicting Claim 8. Thus, $e(Q_1 \bigcap V_2, Q_j) \leq 6$ for every $3 \leq j \leq m$. Hence we have $\sum_{x \in Q_1 \bigcap V_2} d(x, G) \leq 6(k-2) + 9 = 6k - 3$. With the same proof, $\sum_{x \in Q_m \bigcap V_1} d(x, G) \leq 6k - 3$. Therefore, $\sum_{x \in T} d(x, G) \leq 12k - 6$, contradicting the fact $\sum_{x \in T} d(x, G) \geq 12k$. $\square$

Now before completing the whole proof, we will introduce a new chain. Let $L = Q_1Q_2 \dots Q_m$ be the longest directed path in D. Relabel the $m-1$ independent hexagons $Q_1, Q_2, \dots, Q_{m-1}$ such that $d(x_{i,1}, Q_{i+1}) \geq d(x_{i,3}, Q_{i+1}) \geq d(x_{i,2}, Q_{i+1})$ for each $i \in \{1, 2, \dots, m-1\}$. Since $Q_iQ_{i+1} \in A$ for every $1 \leq i \leq m-1$, $e(x_{i,1}x_{i,2}x_{i,3}, Q_{i+1}) \geq 7$. Hence $d(x_{i,1}, Q_{i+1}) = 3$ and $d(x_{i,3}, Q_{i+1}) \geq 2$ for every $1 \leq i \leq m-1$. In particular, since $d(x_{m-1,3}, Q_m) \geq 2$, we have either $x_{m-1,3}y_{m,1} \in E$ or $x_{m-1,3}y_{m,3} \in E$. By symmetry, say $x_{m-1,3}y_{m,1} \in E$. Thus G also contains a new chain, denoted by Chain 2 (see Fig. 3).

By Claim 9, $e(Q_1 \bigcap V_2, Q_m \bigcap V_1) \geq 3$. This implies $m \geq 3$ for otherwise $e(Q_1 \bigcap V_2, Q_m \bigcap V_1) = 0$. Since $Q_mQ_1 \notin A$, we have $e(Q_1 \bigcap V_2, Q_m \bigcap V_1) \leq 6$. Denote $T = (V(Q_1) \bigcap V_2) \bigcup (V(Q_m) \bigcap V_1)$. Since $3 \leq e(Q_1 \bigcap V_2, Q_m \bigcap V_1) \leq 6$, by Claim 9, it follows that either $\exists x \in T$ such that $d(x, T) = 3$ or $G[T]$ contains a quadrilateral. The following proof is divided into two cases.

Case 1. $\exists x \in T$ such that $d(x, T) = 3$.

By symmetry, say $x \in Q_1 \bigcap V_2$. In the following, we will complete our proof using Chain 1 (see Fig. 2). Suppose $d(y_{1,2}, Q_m) = 3$. Since $e(x_{1,1}x_{1,2}x_{1,3}, Q_2) \geq 7$ and $d(x_{1,1}, Q_2) \leq 3$, $e(x_{1,2}x_{1,3}, Q_2) \geq 4$. This implies that $e(x_{1,2}x_{1,3}, y_{2,1}y_{2,2}) \geq 2$, w.l.o.g., say $x_{1,3}y_{2,1} \in E$. So $G[\bigcup_{i=1}^m V(Q_i)]$ contains a 12-cycle $x_{1,2}y_{1,1}x_{1,1}y_{1,3}x_{1,3}y_{2,1}x_{2,1}y_{3,3}x_{2,2}y_{2,2}x_{2,3}y_{2,3}x_{1,2}$. Besides, $G[V(Q_m) - y_{m,3} + y_{1,2}]$ and $G[V(Q_i) - y_{i,3} + y_{i+1,3}]$ $(3 \leq i \leq m-1)$ contain $m-2$ independent hexagons in $G[\bigcup_{i=1}^m V(Q_i)]$. Therefore, $G[\bigcup_{i=1}^m V(Q_i)]$ contains a 12-cycle and $m-2$ hexagons such that all of them are independent. Hence $G \supseteq (k-2)C^6 \bigcup C^{12}$, a contradiction. For the case $d(y_{1,1}, Q_m) = 3$ or $d(y_{1,3}, Q_m) = 3$, we can get a contradiction similarly.

Case 2. $G[T]$ contains a quadrilateral.

In this case, we will complete our discussion using Chain 2 (see Fig. 3). Since $G[T]$ contains a quadrilateral, it follows that $\exists i \neq j, p \neq q$, such that $e(y_{1,i}y_{1,j}, x_{m,p}x_{m,q}) = 4$. We distinguish three subcases.

Subcase 2.1. $\exists p \neq q$, such that $e(y_{1,1}y_{1,3}, x_{m,p}x_{m,q}) = 4$.

Obviously, $d(x_{m,p}, y_{1,1}y_{1,3}) = d(x_{m,q}, y_{1,1}y_{1,3}) = 2$. This implies that there exists $t \in \{1, 2\}$ such that $d(x_{m,t}, y_{1,1}y_{1,3}) = 2$. Hence $G[V(Q_m) \bigcup V(Q_{m-1}) + x_{m-2,1} - x_{m,t}]$ contains a 12-cycle. Besides, $G[V(Q_1) + x_{m,t} - x_{1,1}]$ and $G[V(Q_{i+1}) + x_{i,1} - x_{i+1,1}]$ $(1 \leq i \leq m-3)$ contain $m-2$ independent hexagons in $G[\bigcup_{i=1}^m V(Q_i)]$. Therefore, $G[\bigcup_{i=1}^m V(Q_i)]$ contains a 12-cycle and $m-2$ hexagons such that all of them are independent. Hence $G \supseteq (k-2)C^6 \bigcup C^{12}$, a contradiction.

Subcase 2.2. $\exists p \neq q$, such that $e(y_{1,1}y_{1,2}, x_{m,p}x_{m,q}) = 4$.

Obviously, either $e(y_{1,1}y_{1,2}, x_{m,1}x_{m,2}) = 4$, or $e(y_{1,1}y_{1,2}, x_{m,1}x_{m,3}) = 4$, or $e(y_{1,1}y_{1,2}, x_{m,2}x_{m,3}) = 4$. If $e(y_{1,1}y_{1,2}, x_{m,1}x_{m,2}) = 4$, then $G[V(Q_m) \bigcup \{y_{1,1}, x_{1,2}, y_{1,2}, x_{m-1,1}, y_{m-1,3}, x_{m-1,3}\}]$ contains a 12-cycle $y_{1,1}x_{1,2}y_{1,2}x_{m,1}y_{m,3}x_{m,3}y_{m,2}x_{m-1,1}y_{m-1,3}x_{m-1,3}y_{m,1}x_{m,2}y_{1,1}$. If $e(y_{1,1}y_{1,2}, x_{m,1}x_{m,3}) = 4$, then $G[V(Q_m) \bigcup \{y_{1,1}, x_{1,2}, y_{1,2}, x_{m-1,1}, y_{m-1,3}, x_{m-1,3}\}]$

contains a 12-cycle $y_{1,1}x_{1,2}y_{1,2}x_{m,1}y_{m,3}x_{m-1,1}y_{m-1,3}x_{m-1,3}y_{m,1}x_{m,2}y_{m,2}x_{m,3}y_{1,1}$. If $e(y_{1,1}y_{1,2}, x_{m,2}x_{m,3}) = 4$, then $G[V(Q_m)\bigcup\{y_{1,1}, x_{1,2}, y_{1,2}, x_{m-1,1}, y_{m-1,3}, x_{m-1,3}\}]$ contains a 12-cycle $y_{1,1}x_{1,2}y_{1,2}x_{m,3}y_{m,3}x_{m,1}y_{m,1}x_{m-1,3}y_{m-1,3}x_{m-1,1}y_{m,2}$ $x_{m,2}y_{1,1}$. In each case, $G[V(Q_m)\bigcup\{y_{1,1}, x_{1,2}, y_{1,2}, x_{m-1,1}, y_{m-1,3}, x_{m-1,3}\}]$ contains a 12-cycle. From the definition of Chain 2, $d(x_{i,3}, Q_{i+1}) \geq 2$ for every $1 \leq i \leq m-1$. So $d(x_{i,3}, y_{i+1,1}y_{i+1,2}) \geq 1$. Hence, $e(x_{i,1}x_{i,3}, y_{i+1,1}y_{i+1,2}) \geq 3$, which implies that $G[x_{i,1}, y_{i,3}, x_{i,3}, y_{i+1,1}, x_{i+1,2}, y_{i+1,2}]$ contains a hexagon for every $1 \leq i \leq m-2$. Therefore, $G[\bigcup_{i=1}^{m} V(Q_i)]$ contains a 12-cycle and $m-2$ hexagons such that all of them are independent. Hence $G \supseteq (k-2)C^6 \bigcup C^{12}$, a contradiction.

Subcase 2.3. $\exists p \neq q$, such that $e(y_{1,2}y_{1,3}, x_{m,p}x_{m,q}) = 4$.

Suppose $\{y_{2,1}x_{1,3}, y_{2,3}x_{1,3}\} \subseteq E$. Then $x_{1,3}y_{2,1}x_{2,2}y_{2,2}x_{2,3}y_{2,3}x_{1,3}$ is a hexagon in G, which is independent with the $m-3$ independent hexagons contained in $G[V(Q_i) + x_{i-1,1} - x_{i,1}]$ $(3 \leq i \leq m-1)$. By discussion of p, q, obviously $G[V(Q_1)\bigcup V(Q_m) + x_{m-1,1} - x_{1,3}]$ contains a 12-cycle. Therefore, $G[\bigcup_{i=1}^{m} V(Q_i)]$ contains a 12-cycle and $m-2$ hexagons such that all of them are independent. Hence $G \supseteq (k-2)C^6 \bigcup C^{12}$, a contradiction.

Now we may assume $\{y_{2,1}x_{1,3}, y_{2,3}x_{1,3}\} \nsubseteq E$. This implies $d(x_{1,3}, Q_2) \leq 2$. Since $e(Q_2 \bigcap V_2, Q_1 \bigcap V_1) \geq 7$, we have $d(x_{1,2}, Q_2 \bigcap V_2) \geq 2$. Therefore, $d(x_{1,2}, y_{2,1}y_{2,2}) > 0$. So $G[x_{1,1}, y_{1,1}, x_{1,2}, y_{2,1}, x_{2,2}, y_{2,2}]$ contains a hexagon. By the same proof like Subcase 2.2, $G[V(Q_m)\bigcup\{x_{m-1,1}, y_{m-1,3}, x_{m-1,3}, y_{1,2}, x_{1,3}, y_{1,3}\}]$ contains a cycle of order 12 and $G[x_{i,1}, y_{i,3}, x_{i,3}, y_{i+1,1}, x_{i+1,2}, y_{i+1,2}]$ contains a hexagon for every $2 \leq i \leq m-2$. Therefore, $G[\bigcup_{i=1}^{m} V(Q_i)]$ contains a 12-cycle and $m-2$ hexagons such that all of them are independent. Hence $G \supseteq (k-2)C^6 \bigcup C^{12}$, a contradiction. This completes the whole proof of Theorem 1.4.

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